Teorija opšteg angularnog momenta

- $\hat{\vec{K}}$ – opservabla uopštenog angularnog momenta
- Standardni bazis vektorske opservable $\hat{\vec{K}}$ čine ketovi

$$|k, m_k\rangle$$

što je zajednički svojstveni bazis kompatibilnih observabli $\hat{\vec{K}}^2$ i $\hat{K}_z$. Za elemente standardnog bazisa važi

$$\langle k, m_k | k', m_{k'}\rangle = \delta_{kk'}\,\delta_{m_k m_{k'}}$$

- Za datu vrednost kvantnog broja k, m_k uzima vrednosti iz skupa

$$\{-k, -k+1, \ldots\ k-1, k\}$$

Vrednosti za k mogu biti ili $k = 0, 1, 2, \ldots$ ili $k = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$

- Pod Klebš-Gordanovim koeficijentima podrazumevaju se konstante u razvoju standardnog bazisa opservable $\hat{\vec{K}}$ $(\hat{\vec{K}} = \hat{\vec{K}}_1 + \hat{\vec{K}}_2)$, po bazisu koji nastaje kao direktan proizvod standardnih bazisa za opservable $\hat{\vec{K}}_1$ i $\hat{\vec{K}}_2$, tj:

$$|km\rangle = \sum_{k_1, k_2, m_1} C(km | k_1 k_2 m_1, m_2 = m - m_1)\ |k_1 m_1\rangle \otimes |k_2 m_2\rangle$$

– CG koeficijenti su realni brojevi, što ima za posledicu da obrnuto razlaganje ima oblik:

$$|k_1 m_1\rangle \otimes |k_2 m_2\rangle = \sum_{k \in |k_1-k_2|}^{k_1+k_2} C(km \mid k_1 k_2 m_1, m_2 = m - m_1)\, |km\rangle$$

CG koef. su jednaki nuli ako

1) $m \neq m_1 + m_2$ (ovo je posledica zakona održanja $\vec{J}$)

2) $k \notin \{ |k_1 - k_2|, |k_1 - k_2| + 1, \dots k_1 + k_2 \}$

1. Zadata je vektorska opservabla $\vec{K}$ za čije komponente važe sledeće komutacione relacije:

$[\hat{\vec{K}}^2, \hat{K}_i] = 0, \forall i$; $[\hat{K}_i, \hat{K}_j] = i\hbar \varepsilon_{ijk} \hat{K}_k$. Na osnovu toga naći komutatore:

a) $[\hat{\vec{K}}^2, \hat{K}_\pm]$ c) $[\hat{K}_\pm, \hat{K}_i]$, $i = x, y, z$

b) $[\hat{K}_+, \hat{K}_-]$

gde je $\hat{K}_+ = \hat{K}_x + i\hat{K}_y$ i $\hat{K}_- = \hat{K}_x - i\hat{K}_y$.

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a) $[\hat{\vec{K}}^2, \hat{K}_\pm] = [\hat{\vec{K}}^2, \hat{K}_x \pm i\hat{K}_y] = [\hat{\vec{K}}^2, \hat{K}_x] \pm i[\hat{\vec{K}}^2, \hat{K}_y] =$

$= 0$

b) $[\hat{K}_+, \hat{K}_-] = [\hat{K}_x + i\hat{K}_y, \hat{K}_x - i\hat{K}_y] =$

$= [\hat{K}_x, \hat{K}_x - i\hat{K}_y] + i[\hat{K}_y, \hat{K}_x - i\hat{K}_y] =$

$= [\hat{K}_x, \hat{K}_x] - i[\hat{K}_x, \hat{K}_y] + i[\hat{K}_y, \hat{K}_x] + [\hat{K}_y, \hat{K}_y] =$

$= -2i[\hat{K}_x, \hat{K}_y] = -2i(i\hbar)\underbrace{\varepsilon_{xyz}}_{1}\hat{K}_z$

$= 2\hbar\hat{K}_z$

c) $[\hat{K}_+, \hat{K}_x] = [\hat{K}_x + i\hat{K}_y, \hat{K}_x] = i[\hat{K}_y, \hat{K}_x] =$

$= i(i\hbar)\varepsilon_{yxz}\hat{K}_z = \hbar\hat{K}_z$

$$[\hat{K}_-, \hat{K}_x] = [\hat{K}_x - i\hat{K}_y, \hat{K}_x] = [\hat{K}_x, \hat{K}_x] - i[\hat{K}_y, \hat{K}_x]$$

$$= -i(i\hbar)\,\varepsilon_{yxz} K_z = -\hbar \hat{K}_z$$

$$[\hat{K}_+, \hat{K}_y] = [\hat{K}_x + i\hat{K}_y, \hat{K}_y] = [\hat{K}_x, \hat{K}_y] + i[\hat{K}_y, \hat{K}_y]$$

$$= i\hbar\,\varepsilon_{xyz}\hat{K}_z = i\hbar \hat{K}_z$$

$$[\hat{K}_-, \hat{K}_y] = [\hat{K}_x - i\hat{K}_y, \hat{K}_y] = [\hat{K}_x, \hat{K}_y] - i[\hat{K}_y, \hat{K}_y]$$

$$= i\hbar \hat{K}_z$$

$$[\hat{K}_+, \hat{K}_z] = [\hat{K}_x + i\hat{K}_y, \hat{K}_z] = [\hat{K}_x, \hat{K}_z] + i[\hat{K}_y, \hat{K}_z]$$

$$= i\hbar\,\varepsilon_{xzy}\hat{K}_y + i(i\hbar)\,\varepsilon_{yzx}\hat{K}_x$$

$$= -i\hbar\hat{K}_y - \hbar\hat{K}_x = -\hbar(\hat{K}_x + i\hat{K}_y) = -\hbar\hat{K}_+$$

$$[\hat{K}_-, \hat{K}_z] = [\hat{K}_x - i\hat{K}_y, \hat{K}_z] \overset{\text{Domácí}}{=} \ldots = \hbar \hat{K}_-$$

2. Naći način delovanja operatora $\hat{K}_\pm$ na simultane svojstvene vektore opservabli $\hat{\vec{K}}^2$ i $\hat{K}_z$.

$$\hat{\vec{K}}^2 |km\lambda\rangle = k(k+1)\hbar^2 |km\lambda\rangle \qquad (*)$$

TEORIJA

$$\hat{K}_z |km\lambda\rangle = m\hbar |km\lambda\rangle \qquad (**)$$

λ - dodatni kvantni broj: fiksirano

$$\hat{K}_+ |km\lambda\rangle = ? \quad \text{odnosno}$$

$$\hat{K}_+ |km\rangle = ? \quad \text{Neka je } \hat{K}_+ |km\rangle = |\chi_{km}\rangle$$

$$\hat{K}_z \hat{K}_+ |km\rangle = \hat{K}_z |\chi_{km}\rangle \qquad (1)$$

$$[\hat{K}_+, \hat{K}_z] = -\hbar \hat{K}_+ \Rightarrow \hat{K}_z\hat{K}_+ - \hat{K}_+\hat{K}_z = \hbar \hat{K}_+ \Rightarrow$$

$$\hat{K}_z\hat{K}_+ = \hat{K}_+\hat{K}_z + \hbar \hat{K}_+ = \hat{K}_+(\hat{K}_z + \hbar)$$

L.S. (1)

$$\hat{K}_z\hat{K}_+ |km\rangle = \hat{K}_+(\hat{K}_z + \hbar)|km\rangle = \hat{K}_+(m+1)\hbar |km\rangle =$$

$$= (m+1)\hbar |\chi_{km}\rangle \quad \text{odnosno, zbog (1)}$$

$$\hat{K}_z |\chi_{km}\rangle = (m+1)\hbar |\chi_{km}\rangle \quad \text{a iz } (**)$$

$$\hat{K}_z |km\rangle = m\hbar |km\rangle \quad \text{i poređenjem ova dva}$$

$$|\chi_{km}\rangle = C(k,m)\,|k\ m+1\rangle \qquad (2)$$

Setiti se da je

$$\hat{K}_+|km\rangle = |\chi_{km}\rangle \Rightarrow \langle km|\hat{K}_- = \langle\chi_{km}| \qquad (3)$$

Iz (3) $\Rightarrow$

$$\langle\chi_{km}|\chi_{km}\rangle = |C(k,m)|^2$$ odnosno

$$\langle km|\hat{K}_-\hat{K}_+|km\rangle = |C(k,m)|^2$$

$$\hat{K}_-\hat{K}_+ = (\hat{K}_x - i\hat{K}_y)(\hat{K}_x + i\hat{K}_y) =$$

$$= \hat{K}_x^2 + i\hat{K}_x\hat{K}_y - i\hat{K}_y\hat{K}_x + \hat{K}_y^2 =$$

$$= \hat{K}_x^2 + \hat{K}_y^2 + i[\hat{K}_x,\hat{K}_y] = \hat{K}_x^2 + \hat{K}_y^2 + i(i\hbar)\varepsilon_{xyz}\hat{K}_z$$

$$= \hat{K}_x^2 + \hat{K}_y^2 - \hbar\hat{K}_z = \hat{\vec{K}}^2 - \hat{K}_z^2 - \hbar\hat{K}_z$$

$$\langle km|\hat{\vec{K}}^2 - \hat{K}_z^2 - \hbar\hat{K}_z|km\rangle = |C(k,m)|^2$$

$$\langle km|k(k+1)\hbar^2 - m^2\hbar^2 - \hbar^2 m|km\rangle = |C(k,m)|^2$$

$$C(k,m) = \hbar\sqrt{k(k+1) - m(m+1)}$$

odnosno

$$\hat{K}_+|km\rangle = |\chi_{km}\rangle \overset{(2)}{=} \hbar\sqrt{k(k+1)-m(m+1)}\;|k\ m+1\rangle$$

$K_- |km\rangle = ?$

Neka je $\hat{K}_- |km\rangle = |\varphi_{km}\rangle$

$\Big\lceil\; [\hat{K}_-, \hat{K}_z] = \hbar \hat{K}_- \Rightarrow \hat{K}_z \hat{K}_- = \hat{K}_- \hat{K}_z - \hbar \hat{K}_- =$

$= \hat{K}_- (\hat{K}_z - \hbar) \;\Big\rfloor$

$$\hat{K}_z \hat{K}_- |km\rangle = \hat{K}_z |\varphi_{km}\rangle \qquad (4)$$

$\hat{K}_z |\varphi_{km}\rangle = \hat{K}_- (\hat{K}_z - \hbar) |km\rangle = \hat{K}_- (m-1)\hbar |km\rangle$

$= (m-1)\hbar \hat{K}_- |km\rangle = (m-1)\hbar |\varphi_{km}\rangle$, tj.

$\hat{K}_z |\varphi_{km}\rangle = (m-1)\hbar |\varphi_{km}\rangle$ a iz (**)

$\hat{K}_z |km\rangle = m\hbar |km\rangle$ i poređenjem ova dva

$$|\varphi_{km}\rangle = D(k,m) |k\, m-1\rangle \qquad (5)$$

$$\hat{K}_- |km\rangle = |\varphi_{km}\rangle \Rightarrow \langle km| \hat{K}_+ = \langle \varphi_{km}| \qquad (6)$$

Iz (6) $\Rightarrow$

$\langle \varphi_{km} | \varphi_{km}\rangle = \langle km | \hat{K}_+ \hat{K}_- | km\rangle$

$\Big\lceil\; \hat{K}_+ \hat{K}_- = (\hat{K}_x + i\hat{K}_y)(\hat{K}_x - i\hat{K}_y) = \hat{K}_x^2 - i\hat{K}_x\hat{K}_y + i\hat{K}_y\hat{K}_x + \hat{K}_y^2 =$

$= \hat{K}_x^2 + \hat{K}_y^2 + i[\hat{K}_y, \hat{K}_x] = \hat{\vec{K}}^2 - \hat{K}_z^2 + i(i\hbar)\varepsilon_{yxz}\hat{K}_z = \hat{\vec{K}}^2 - \hat{K}_z^2 + \hbar\hat{K}_z \;\Big\rfloor$

$\langle \varphi_{km} | \varphi_{km} \rangle = \langle km | \hat{K}^2 - \hat{K}_z^2 + \hbar \hat{K}_z | km \rangle$

$= \hbar^2 [(k+1)k - m(m+1)]$, odnosno zbog (5)

$\hbar^2 [(k+1)k - m(m-1)] = |D(k,m)|^2 \Rightarrow$

$D(k,m) = \hbar \sqrt{k(k+1) - m(m-1)}$

Konačno

$\hat{K}_- |km\rangle = |\varphi_{km}\rangle = \hbar \sqrt{k(k+1) - m(m-1)} \, |k\, m-1\rangle$

Bitno za dalji rad

$\hat{K}_+ |km\rangle = \hbar \sqrt{k(k+1) - m(m+1)} \, |k\, m+1\rangle$

$\hat{K}_- |km\rangle = \hbar \sqrt{k(k-1) - m(m-1)} \, |k\, m-1\rangle$

3. Rešiti svojstveni problem z-komponente vektorske opservable $\vec{\hat{\sigma}}$ u dvodimenzionalnom vektorskom prostoru i na osnovu toga naći reprezentaciju komponenata vektorske opservable u $\hat{\sigma}_z$-reprezentaciji. Komponente vektorske opservable $\vec{\hat{\sigma}}$ zadovoljavaju izraz

$$\hat{\sigma}_i \hat{\sigma}_j = \delta_{ij} \hat{I} + i \varepsilon_{ijk} \hat{\sigma}_k \qquad (*)$$

Iz (*) $\hat{\sigma}_i^2 = \hat{I}$, $\forall i$ i

sledeće jednakosti

$$\hat{\sigma}_x \hat{\sigma}_y = i \hat{\sigma}_z, \quad \hat{\sigma}_y \hat{\sigma}_z = i \hat{\sigma}_x, \quad \hat{\sigma}_z \hat{\sigma}_x = i \hat{\sigma}_y$$

$$\hat{\sigma}_y \hat{\sigma}_x = -i \hat{\sigma}_z, \quad \hat{\sigma}_z \hat{\sigma}_y = -i \hat{\sigma}_x, \quad \hat{\sigma}_x \hat{\sigma}_z = -i \hat{\sigma}_y$$

$\hat{\sigma}_i^\dagger = \hat{\sigma}_i$, $\forall i$ što sledi iz opšte teorije angularnog momenta

Svojstveni problem $\hat{\sigma}_z$

$$\hat{\sigma}_z |\sigma_z\rangle = \sigma_z |\sigma_z\rangle$$

$$\underbrace{\hat{\sigma}_z^2}_{\hat{I}} |\sigma_z\rangle = \sigma_z \sigma_z |\sigma_z\rangle = \sigma_z^2 |\sigma_z\rangle$$

$$\sigma_z^2 |\sigma_z\rangle = |\sigma_z\rangle \Rightarrow \sigma_z^2 = 1 \Rightarrow \sigma_z \begin{cases} 1 \\ -1 \end{cases}$$

$$\hat{\sigma}_z |1\rangle = |1\rangle$$

$$\hat{\sigma}_z |-1\rangle = -1 |-1\rangle$$

$$\hat{\sigma}_z \rightarrow \begin{pmatrix} \langle 1|\hat{\sigma}_z|1\rangle & \langle 1|\hat{\sigma}_z|-1\rangle \\ \langle -1|\hat{\sigma}_z|1\rangle & \langle -1|\hat{\sigma}_z|-1\rangle \end{pmatrix}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

σ_x матрица

$$\sigma_x = \begin{pmatrix} \langle 1|\hat{\sigma}_x|1\rangle & \langle 1|\hat{\sigma}_x|-1\rangle \\ \langle -1|\hat{\sigma}_x|1\rangle & \langle -1|\hat{\sigma}_x|-1\rangle \end{pmatrix}$$

Prvo dijagonalni elementi

$$\langle 1|\hat{\sigma}_x|1\rangle = \langle 1|\, i\hat{\sigma}_z\hat{\sigma}_y|1\rangle = i\langle 1|\hat{\sigma}_y|1\rangle$$

$$\langle 1|\hat{\sigma}_y|1\rangle = \langle 1|\, i\hat{\sigma}_x\hat{\sigma}_z|1\rangle = i\langle 1|\hat{\sigma}_x|1\rangle$$

$$\langle 1|\hat{\sigma}_x|1\rangle = i^2\langle 1|\hat{\sigma}_x|1\rangle = -\langle 1|\hat{\sigma}_x|1\rangle \Rightarrow$$

$$\langle 1|\hat{\sigma}_x|1\rangle = 0$$

Slično tome, dobija se da važi

$$\langle -1|\hat{\sigma}_x|-1\rangle = 0$$

Ostaju vandijagonalni elementi

$$\hat{S}_z = \frac{\hbar}{2}\hat{\sigma}_z \Rightarrow$$

$$\left.\begin{aligned} \hat{S}_z|1\rangle &= \frac{\hbar}{2}|1\rangle \\ \hat{S}_z|-1\rangle &= -\frac{\hbar}{2}|-1\rangle \end{aligned}\right\}$$

Poređenjem sa opštom relacijom

$$\hat{K}_z|km\rangle = m\hbar|km\rangle$$

Zaključuje se da je $m = \pm\frac{1}{2}$ a $s = \frac{1}{2}$.

Dakle

$$|1\rangle = \left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle \quad , \quad |-1\rangle = \left|\tfrac{1}{2}\,-\tfrac{1}{2}\right\rangle$$

$$\left.\begin{aligned} \hat{S}_\pm &= \hbar\sqrt{s(s+1)-m(m\pm1)}\;|s\; m\pm1\rangle \\ \hat{S}_\pm &= \frac{\hbar}{2}\hat{\sigma}_\pm \end{aligned}\right\} \Rightarrow$$

$$\hat{\sigma}_\pm|sm\rangle = 2\sqrt{s(s+1)-m(m\pm1)}\;|s\; m\pm1\rangle \qquad (**)$$

$$\langle -1|\hat{\sigma}_x|1\rangle = \langle -1|\frac{\hat{\sigma}_+ + \hat{\sigma}_-}{2}|1\rangle =$$

$$= \left\langle \tfrac{1}{2}\, -\tfrac{1}{2}\right|\frac{\hat{\sigma}_+ + \hat{\sigma}_-}{2}\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle = \frac{1}{2}\left\langle \tfrac{1}{2}\, -\tfrac{1}{2}\right|\hat{\sigma}_+\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle +$$

$$+\frac{1}{2}\left\langle \tfrac{1}{2}\, -\tfrac{1}{2}\right|\hat{\sigma}_-\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle = \frac{1}{2}\left\langle \tfrac{1}{2}\, -\tfrac{1}{2}\middle|\tfrac{1}{2}\,\tfrac{3}{2}\right\rangle 2\sqrt{\tfrac{1}{2}\left(\tfrac{1}{2}+1\right)-\tfrac{1}{2}\left(\tfrac{1}{2}+1\right)}$$

$$+\frac{1}{2}\left\langle \tfrac{1}{2}\, -\tfrac{1}{2}\middle|\tfrac{1}{2}\, -\tfrac{1}{2}\right\rangle 2\sqrt{\tfrac{1}{2}\left(\tfrac{1}{2}+1\right)-\tfrac{1}{2}\left(\tfrac{1}{2}-1\right)} =$$

$$\sqrt{\frac{3}{4}+\frac{1}{4}} = 1$$

Drugi vandijagonalni element

$$\langle 1|\hat{\sigma}_x|-1\rangle = \left(\langle -1|\hat{\sigma}_x|1\rangle\right)^*$$

$$\hat{\sigma}_x \longrightarrow \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Za vezbu

$$\hat{\sigma}_y \rightarrow \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

4. Zadata je vektorska opservabla $\hat{\vec{J}}$ izrazom $\hat{\vec{J}} = \hat{\vec{L}} + \hat{\vec{S}}$ gde je $\hat{\vec{L}}$ opservabla momenta impulsa a $\hat{\vec{S}}$ je opservabla spina. Dokazati da kompo- nente opservable $\hat{\vec{J}}$ zadovoljavaju algebru opšteg angularnog momenta $\hat{\vec{K}}$ i na osnovi toga naći vezu između kvantnih brojeva j sa jedne i l i s sa druge strane, tj. vezu između m_j sa jedne i m_l i m_s sa druge strane.

$$\hat{\vec{L}}, \hat{\vec{S}} \longrightarrow \hat{\vec{J}} = \hat{\vec{L}} + \hat{\vec{S}} \equiv \hat{L} \otimes \hat{I}_s + \hat{I}_L \otimes \hat{S}$$

$$\hat{\vec{L}}^2 |l m_l \lambda\rangle = l(l+1)\hbar^2 |l m_l \lambda\rangle$$

$$\hat{L}_z |l m_l \lambda\rangle = m_l \hbar |l m_l \lambda\rangle$$

$$\hat{\vec{S}}^2 |s m_s \lambda\rangle = s(s+1)\hbar^2 |s m_s \lambda\rangle$$

$$\hat{S}_z |s m_s \lambda\rangle = m_s \hbar |s m_s \lambda\rangle$$

$$\left.\begin{aligned} [\hat{\vec{K}}^2, \hat{K}_i] &= 0 \\ [\hat{K}_i, \hat{K}_j] &= i\hbar \varepsilon_{ijk} \hat{K}_k \end{aligned}\right\} \text{ ove relacije pokazati za } \hat{\vec{J}}$$

$$[\hat{\vec{J}}^2, \hat{J}_i] = ?$$

$$[\hat{\vec{J}}^2, \hat{J}_i] = [(\hat{\vec{L}} + \hat{\vec{S}})^2, \hat{L}_i + \hat{S}_i] = [\hat{\vec{L}}^2 + \hat{\vec{S}}^2 + 2\hat{\vec{L}}\cdot\hat{\vec{S}}, \hat{L}_i + \hat{S}_i]$$

$$= [\hat{\vec{L}}^2, \hat{L}_i + \hat{S}_i] + [\hat{\vec{S}}^2, \hat{L}_i + \hat{S}_i] + 2[\hat{\vec{L}}\cdot\hat{\vec{S}}, \hat{L}_i + \hat{S}_i] =$$

$$= \cancelto{0}{[\hat{\vec{L}}^2, \hat{L}_i]} + \cancelto{0}{[\hat{\vec{L}}^2, \hat{S}_i]} + \cancelto{0}{[\hat{\vec{S}}^2, \hat{L}_i]} + \cancelto{0}{[\hat{\vec{S}}^2, \hat{S}_i]}$$

$$+ 2[\hat{\vec{L}}\cdot\hat{\vec{S}}, \hat{L}_i] + 2[\hat{\vec{L}}\cdot\hat{\vec{S}}, \hat{S}_i] =$$

$$2\left([\hat{L}_j\hat{S}_j, \hat{L}_i] + [\hat{L}_j\hat{S}_j, \hat{S}_i]\right) = 0$$

$$[\hat{L}_j\hat{S}_j, \hat{L}_i] = [\hat{L}_j, \hat{L}_i]\hat{S}_j + \hat{L}_j \cancelto{0}{[\hat{S}_j, \hat{L}_i]}$$

$$= i\hbar\, \varepsilon_{jik}\, \hat{L}_k \hat{S}_j \qquad (*)$$

$$[\hat{L}_j\hat{S}_j, \hat{S}_i] = \cancelto{0}{[\hat{L}_j, \hat{S}_i]}\hat{S}_j + \hat{L}_j [\hat{S}_j, \hat{S}_i]$$

$$= \hat{L}_j\, i\hbar\, \varepsilon_{jim}\, \hat{S}_m$$

$$= i\hbar\, \varepsilon_{jim}\, \hat{L}_j \hat{S}_m \qquad (**)$$

Da bi uporediti (*) i (**) nema u (**)

$j \to k$, $m \to j$

pa će biti $i\hbar\, \varepsilon_{kij}\, \hat{L}_k \hat{S}_j = -i\hbar\, \varepsilon_{jik}\, \hat{L}_k \hat{S}_j$ a to je ~~negativnog~~ suprotnog predznaka u odnosu na (*)

kij
-ikj
ijk

~~jik~~
jik

kij
-ikj
ijk
-jik

jik = -kij

Dakle

$$[\hat{\vec{J}}^2, \hat{J}_i] = 0$$

Dalje, $[J_i, J_l] = ?$

$$[\hat{J}_i, \hat{J}_j] = [\hat{L}_i + \hat{S}_i, \hat{L}_j + \hat{S}_j] = [\hat{L}_i, \hat{L}_j] + \cancel{[\hat{L}_i, \hat{S}_j]} + \cancel{[\hat{S}_i, \hat{L}_j]} + [\hat{S}_i, \hat{S}_j] = i\hbar\, \varepsilon_{ijk} \hat{L}_k + i\hbar\, \varepsilon_{ijk} \hat{S}_k$$

$$= i\hbar\, \varepsilon_{ijk} (\hat{L}_k + \hat{S}_k) = i\hbar\, \varepsilon_{ijk} \hat{J}_k$$

Zaključak je da za $\hat{\vec{J}}$ mora da važi

$$\hat{\vec{J}}^{\,2} |j\, m_j\, \lambda\rangle = j(j+1)\hbar^2 |j\, m_j\, \lambda\rangle$$

$$\hat{J}_z |j\, m_j\, \lambda\rangle = m_j \hbar |j\, m_j\, \lambda\rangle \implies$$

$$j \in \{ |l-s|, |l-s+1| \ldots |l+s| \}$$

$$m_j = m_l + m_s$$

5. Naći simultana svojstvena stanja (opservabli $\hat{\vec{L}}^2$) $\hat{\vec{L}}^2$ i $\hat{L}_x$ koja odgovaraju svojstvenim vrednostima $2\hbar^2$ i $\hbar$, redom, u $(\hat{\vec{L}}^2, \hat{L}_z)$ reprezentaciji.

$$\hat{\vec{L}}^2 |l m_l\rangle_x = l(l+1)\hbar^2 |l m_l\rangle_x$$

$$l(l+1)\hbar^2 = 2\hbar^2 \Rightarrow l=1$$

$l=1 \Rightarrow m_l = \{-1, 0, 1\}$ ali $\hat{L}_x |l m_l\rangle_x = m_l\hbar |l m_l\rangle_x$

$m_l\hbar = \hbar \Rightarrow m_l = 1$

Dakle $l=1$, $m_l=1$ i stanje je $|11\rangle_x$

u $(\hat{\vec{L}}^2, \hat{L}_z)$ reprezentaciji

$$\hat{\vec{L}}^2 |1 m_l\rangle_z = 2\hbar^2 |1 m_l\rangle_z$$

$$\hat{L}_z |1 m_l\rangle_z = m_l\hbar |1 m_l\rangle_z$$

jer $[\hat{L}_z, \hat{L}_x] \neq 0$

ovo nije definitivno određeno pa su svojstvena stanja

$|1 -1\rangle_z$, $|1 0\rangle_z$, $|1 +1\rangle_z$

$$|11\rangle_x = c_{-1} |1 -1\rangle_z + c_0 |10\rangle_z + c_1 |1 +1\rangle_z \qquad (*)$$

$$\boxed{c_{-1}, c_0, c_1 = ?}$$

Koristimo

$$\hat{L}_x |11\rangle_x = \hbar \; |11\rangle_x \qquad (**)$$

$$\hat{L}_x = \frac{1}{2} (\hat{L}_+ + \hat{L}_-) \qquad (***)$$

L.S. od $(*)$

$$\hat{L}_x |11\rangle_x = \hbar |11\rangle_x = \hbar \left(c_{-1} |1\,{-1}\rangle_z + c_0 |10\rangle_z + c_1 |1{+}1\rangle_z \right)$$

D.S. od $(*)$

$$\frac{1}{2} (\hat{L}_+ + \hat{L}_-) \left(c_{-1} |1{-}1\rangle_z + c_0 |10\rangle_z + c_1 |1{+}1\rangle_z \right) =$$

$$\frac{1}{2} \Big(c_{-1} \hat{L}_+ |1{-}1\rangle_z + c_0 \hat{L}_+ |10\rangle_z + c_1 \hat{L}_+ |1{+}1\rangle_z +$$
$$c_{-1} \hat{L}_- |1{-}1\rangle_z + c_0 \hat{L}_- |10\rangle_z + c_1 \hat{L}_- |1{+}1\rangle_z \Big) =$$

$$\boxed{\begin{aligned} \hat{L}_+ |\ell m\rangle &= \hbar \sqrt{\ell(\ell+1) - m(m+1)} \; |\ell \; m{+}1\rangle \\ \hat{L}_- |\ell m\rangle &= \hbar \sqrt{\ell(\ell+1) - m(m-1)} \; |\ell \; m{-}1\rangle \end{aligned}}$$

$$\frac{1}{2} \Big(c_{-1} \sqrt{2} \hbar |10\rangle_z + c_0 \sqrt{2} \hbar |11\rangle_z + 0 |12\rangle_z + 0 |1{-}2\rangle_z$$

$$c_0 \hbar \sqrt{2} |1{-}1\rangle_z + c_1 \hbar \sqrt{2} |10\rangle_z \Big) =$$

$$= \frac{1}{2} (c_{-1} + c_1) \hbar \sqrt{2} |10\rangle_z + \frac{c_0 \sqrt{2} \hbar}{2} |11\rangle_z + \frac{c_0 \sqrt{2} \hbar}{2} |1{-}1\rangle_z$$

Upoređivši L.S. i D.S.

$$c_0 = \frac{\sqrt{2}}{2}(c_{-1}+c_1)$$

$$c_1 = \frac{\sqrt{2}}{2}c_0 \qquad (****)$$

$$c_{-1} = \frac{\sqrt{2}}{2}c_0$$

Koeficijenti u (*) moraju da zadovolje uslov normiranja

$$|c_{-1}|^2 + |c_0|^2 + |c_1|^2 = 1$$

$$\frac{1}{2}|c_0|^2 + |c_0|^2 + \frac{1}{2}|c_0|^2 = 1$$

$$2|c_0|^2 = 1 \Rightarrow |c_0| = \frac{\sqrt{2}}{2} \Rightarrow c_0 = \frac{\sqrt{2}}{2}$$

Znajući c_0, smenom u (****) dobija se

$$c_1 = \frac{\sqrt{2}}{2}c_0 = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} = \frac{1}{2}$$

$$c_{-1} = \frac{\sqrt{2}}{2}c_0 = \frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2} = \frac{1}{2}$$

Dakle, stanje $|11\rangle_x$ ima oblik

$$|11\rangle_x = \frac{1}{2}|1\,-1\rangle_z + \frac{\sqrt{2}}{2}|10\rangle_z + \frac{1}{2}|11\rangle_z$$

6. Naći zajedničko svojstvena stanja opservabli $\hat{\vec{L}}^2$ i $\hat{L}_x^2 - \hat{L}_y^2$ koje odgovaraju svojstvenoj vrijednosti $2\hbar^2$ operatora $\hat{\vec{L}}^2$, u $(\hat{\vec{L}}^2, \hat{L}_z)$ reprezentaciji.

$$\hat{\vec{L}}^2 |\psi\rangle = \hbar^2 l(l+1) |\psi\rangle \qquad \boxed{l=1}$$

$$(\hat{L}_x^2 - \hat{L}_y^2)|\psi\rangle = \lambda\hbar^2 |\psi\rangle$$

$$|\psi\rangle = c_{-1}|1\,-1\rangle_z + c_0 |1\,0\rangle_z + c_1 |1\,1\rangle_z \qquad (*)$$

$$[\hat{\vec{L}}^2, \hat{L}_z] = 0$$

$$\left.\begin{aligned} \hat{L}_x &= \frac{1}{2}(\hat{L}_+ + \hat{L}_-) \\ \hat{L}_y &= \frac{1}{2i}(\hat{L}_+ - \hat{L}_-) \end{aligned}\right\} \Rightarrow$$

$$\boxed{\hat{L}_x^2 - \hat{L}_y^2 = \frac{1}{2}(\hat{L}_+^2 + \hat{L}_-^2)}$$

$$(\hat{L}_x^2 - \hat{L}_y^2)|\psi\rangle = \frac{1}{2}(\hat{L}_+^2 + \hat{L}_-^2)|\psi\rangle =$$

$$= \frac{1}{2}(\hat{L}_+^2 + \hat{L}_-^2)\left(c_{-1}|1\,-1\rangle_z + c_0|1\,0\rangle_z + c_1|1\,1\rangle_z\right)$$

$$= \frac{1}{2}\Big[c_{-1}\hat{L}_+^2|1\,-1\rangle_z + c_0\hat{L}_+^2|1\,0\rangle_z + c_1\hat{L}_+^2|1\,1\rangle_z +$$
$$c_{-1}\hat{L}_-^2|1\,-1\rangle_z + c_0\hat{L}_-^2|1\,0\rangle_z + c_1\hat{L}_-^2|1\,1\rangle_z\Big] =$$

$\hat{L}_+^2 |1\,-1\rangle_z = \hat{L}_+ \hat{L}_+ |1\,-1\rangle_z = \hat{L}_+ \left(\hbar\sqrt{2}\; |1\,0\rangle_z \right)$

$= \sqrt{2}\hbar\, \hat{L}_+ |1\,0\rangle_z = \sqrt{2}\hbar\, \sqrt{2}\hbar\, |1\,1\rangle_z = 2\hbar^2 |1\,1\rangle_z$

$$\left.\begin{aligned} &\hat{L}_+^2 |1\,0\rangle_z = 0 \\ &\hat{L}_+^2 |1\,1\rangle_z = 0 \\ &\hat{L}_-^2 |1\,-1\rangle_z = 0 \\ &\hat{L}_-^2 |1\,0\rangle_z = 0 \end{aligned}\right\} \quad \text{Domaći, provera}$$

$\hat{L}_-^2 |1\,1\rangle_z = \dots = 2\hbar^2 |1\,-1\rangle_z$

$= \frac{1}{2}\left[2\hbar^2 c_{-1} |1\,1\rangle_z + 2\hbar^2 c_1 |1\,-1\rangle_z \right]$

$= \hbar^2 \left(c_{-1} |1\,1\rangle_z + c_1 |1\,-1\rangle_z \right)$

Dakle, imamo

$(\hat{L}_x^2 - \hat{L}_y^2)|\psi\rangle = \hbar^2 \left(c_{-1}|1\,1\rangle_z + c_1 |1\,-1\rangle_z \right)$ i

$(\hat{L}_x^2 - \hat{L}_y^2)|\psi\rangle = \lambda\hbar^2 \left(c_{-1} |1\,-1\rangle_z + c_0 |1\,0\rangle_z + c_1 |1\,1\rangle_z \right)$

Poređenjem desnih strana se dobija

$$\left.\begin{aligned} \lambda c_1 &= c_{-1} \\ \lambda c_{-1} &= c_1 \end{aligned}\right\} \quad c_1 = \lambda^2 c_1 \Rightarrow \lambda = \pm 1$$

$\lambda c_0 = 0$

$\lambda = 0 \qquad c_1 = c_{-1} = 0$

$$\left.\begin{matrix} \lambda = 0 \qquad c_1 = c_{-1} = 0 \\ |c_{-1}|^2 + |c_0|^2 + |c_1|^2 = 1 \end{matrix}\right\} \Rightarrow c_0 = 1$$

$|\psi\rangle = |10\rangle_z$

$\lambda = 1$

$$\left.\begin{matrix} c_0 = 0 \qquad c_1 = c_{-1} \\ |c_{-1}|^2 + |c_0|^2 + |c_1|^2 = 1 \end{matrix}\right\} \Rightarrow c_1 = c_{-1} = \frac{1}{\sqrt{2}}$$

$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|1 -1\rangle_z + |11\rangle_z\right)$

$\lambda = -1$

$$\left.\begin{matrix} c_0 = 0 \qquad c_1 = -c_{-1} \\ |c_{-1}|^2 + |c_0|^2 + |c_1|^2 = 1 \end{matrix}\right\} \Rightarrow c_1 = -c_{-1} = \frac{1}{\sqrt{2}}$$

$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|1 -1\rangle_z - |11\rangle_z\right)$

Za ispit:

a) $\hat{\vec{L}}^2, \; \hat{L}_x^2 + \hat{L}_y^2$

b) $\hat{\vec{L}}^2, \; \hat{L}_y^2$

c) $\hat{\vec{L}}^2, \; \hat{L}_x^2$

7. Dati su spinovi definisani kvantnim brojevima $s_1 = s_2 = \frac{1}{2}$. Naći Klebš-Gordanove koeficijente za sva stanja u standardnom bazisu opservable $\hat{\vec{S}} = \hat{\vec{S}}_1 + \hat{\vec{S}}_2$.

$\hat{\vec{S}}_1 \leftrightarrow \mathcal{H}^{(1)}$ $\qquad$ $\hat{\vec{S}}_2 \leftrightarrow \mathcal{H}^{(2)}$

$s_1 = \frac{1}{2}$ $\qquad$ $s_2 = \frac{1}{2}$

$m_1 = \pm\frac{1}{2}$ $\qquad$ $m_2 = \pm\frac{1}{2}$

$$\left.\begin{aligned} \hat{\vec{S}}^2 |sm\rangle &= \hbar^2 s(s+1)|sm\rangle \\ \hat{S}_z |sm\rangle &= m\hbar |sm\rangle \end{aligned}\right\} \hat{\vec{S}} \leftrightarrow \mathcal{H} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$$

$$s \in \{ |s_1 - s_2|, |s_1 - s_2| + 1, \dots s_1 + s_2 \} = \{0, 1\}$$

$s = 0, \; m = 0$

$s = 1, \; m = 0, \pm 1$

Standardni bazis za opservablu $\hat{\vec{S}}$

$$\{|sm\rangle\} = \{ |00\rangle, |11\rangle, |10\rangle, |1-1\rangle \}$$

ONB u $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$

$|\frac{1}{2}\frac{1}{2}\rangle_1 \otimes |\frac{1}{2}\frac{1}{2}\rangle_2$

$|\frac{1}{2}\frac{1}{2}\rangle_1 \otimes |\frac{1}{2}-\frac{1}{2}\rangle_2$

$|\frac{1}{2}-\frac{1}{2}\rangle_1 \otimes |\frac{1}{2}\frac{1}{2}\rangle_2$

$|\frac{1}{2}-\frac{1}{2}\rangle_1 \otimes |\frac{1}{2}-\frac{1}{2}\rangle_2$

$$|sm\rangle = \sum_{s_1, s_2, m_1} C(s_1 s_2\, m_1, m_2 = m - m_1 ;\, sm)\, |s_1 m_1\rangle_1 \otimes |s_2 m_2\rangle_2$$

CG koeficijenti $= 0$ ako je $m \neq m_1 + m_2$

$$|11\rangle = c_1 \left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle_1 \otimes \left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle_2 + c_2 \left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle_1 \otimes \left|\tfrac{1}{2}-\tfrac{1}{2}\right\rangle_2 +$$

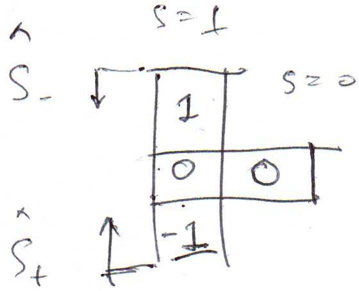
$$+ c_3 \left|\tfrac{1}{2}-\tfrac{1}{2}\right\rangle_1 \otimes \left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle_2 + c_4 \left|\tfrac{1}{2}-\tfrac{1}{2}\right\rangle_1 \otimes \left|\tfrac{1}{2}-\tfrac{1}{2}\right\rangle_2$$

$c_2 = c_3 = c_4 = 0$

$$|11\rangle = c_1 \left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle_1 \otimes \left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle_2$$

$\langle 11|11\rangle = 1 \Rightarrow c_1 = 1$

$$\boxed{|11\rangle = \left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle_1 \otimes \left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle_2}$$



Slično i

$$\boxed{|1\,{-1}\rangle = \left|\tfrac{1}{2}-\tfrac{1}{2}\right\rangle_1 \otimes \left|\tfrac{1}{2}-\tfrac{1}{2}\right\rangle_2}$$

$$|10\rangle = c_1 \left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle_1 \otimes \left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle_2 + c_2 \left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle_1 \otimes \left|\tfrac{1}{2}-\tfrac{1}{2}\right\rangle_2 +$$

$$c_3 \left|\tfrac{1}{2}-\tfrac{1}{2}\right\rangle_1 \otimes \left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle_2 + c_4 \left|\tfrac{1}{2}-\tfrac{1}{2}\right\rangle_1 \otimes \left|\tfrac{1}{2}-\tfrac{1}{2}\right\rangle_2$$

$c_1 = c_4 = 0$

$$|10\rangle = c_2 \left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle_1 \otimes \left|\tfrac{1}{2}-\tfrac{1}{2}\right\rangle_2 + c_3 \left|\tfrac{1}{2}-\tfrac{1}{2}\right\rangle_1 \otimes \left|\tfrac{1}{2}-\tfrac{1}{2}\right\rangle_2$$

$$S_-|11\rangle = (S_{1-}\otimes I_2 + I_1 \otimes \hat{S}_{2-})\,|\tfrac{1}{2}\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle_2$$

$$\hat{S}_-|11\rangle = \hbar\sqrt{1(1+1)-1(1-1)}\;|10\rangle = \sqrt{2}\hbar\,|10\rangle$$

$$(\hat{S}_{1-}\otimes\hat{I}_2 + \hat{I}_1\otimes\hat{S}_{2-})\,|\tfrac{1}{2}\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle_2 =$$

$$= (\hat{S}_{1-}\otimes\hat{I}_2)\,|\tfrac{1}{2}\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle_2 + (\hat{I}_1\otimes\hat{S}_{2-})\,|\tfrac{1}{2}\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle_2 =$$

$$\left[\; \hat{S}_{1-}|\tfrac{1}{2}\tfrac{1}{2}\rangle_1 = \hbar\sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1)-\tfrac{1}{2}(\tfrac{1}{2}-1)}\;|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_1 \right.$$

$$= \hbar\sqrt{\tfrac{3}{4}+\tfrac{1}{4}}\;|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle = \hbar\,|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_1$$

$$\left. \hat{S}_{2-}|\tfrac{1}{2}\tfrac{1}{2}\rangle_2 = \hbar\,|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_2 \;\right]$$

$$= \hbar\,|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle_2 + \hbar\,|\tfrac{1}{2}\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_2$$

Odnosno

$$\sqrt{2}\hbar\,|10\rangle = \hbar\left(|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle_2 + |\tfrac{1}{2}\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_2\right)$$

$$\boxed{|10\rangle = \frac{1}{\sqrt{2}}\left(|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle_2 + |\tfrac{1}{2}\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_2\right)}$$

$$|00\rangle = c_2\,|\tfrac{1}{2}\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_2 + c_3\,|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2}\,+\tfrac{1}{2}\rangle_2$$

Delujemo operatorom $\hat{S}_+ = \hat{S}_{1+}\otimes\hat{I}_2 + \hat{I}_1\otimes\hat{S}_{2+}$ na LS i DS gornje jednakosti, respektivno.

$\hat{S}_+ |00\rangle = 0$ (proveriti!)

$$\hat{S}_{1+} |\tfrac{1}{2}\tfrac{1}{2}\rangle_1 = \hbar \sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1) - \tfrac{1}{2}(\tfrac{1}{2}+1)}\ |\tfrac{1}{2}\tfrac{3}{2}\rangle_1 = 0$$

$$\hat{S}_{1+} |\tfrac{1}{2}-\tfrac{1}{2}\rangle_1 = \hbar \sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1) - \tfrac{1}{2}(-\tfrac{1}{2}+1)}\ |\tfrac{1}{2}\tfrac{1}{2}\rangle_1$$

$$= \hbar \sqrt{\tfrac{3}{4} - \tfrac{1}{4}}\ |\tfrac{1}{2}\tfrac{1}{2}\rangle_1 = \frac{\hbar}{\sqrt{2}} |\tfrac{1}{2}\tfrac{1}{2}\rangle_1$$

$$\hat{S}_{2+} |\tfrac{1}{2}-\tfrac{1}{2}\rangle_2 = \frac{\hbar}{\sqrt{2}} |\tfrac{1}{2}\tfrac{1}{2}\rangle_2$$

$$\hat{S}_{2+} |\tfrac{1}{2}\tfrac{1}{2}\rangle_2 = 0$$

Dakle,

$$0 = c_3 \frac{\hbar}{\sqrt{2}} |\tfrac{1}{2}\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle_2 + c_2 \frac{\hbar}{\sqrt{2}} |\tfrac{1}{2}\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle_2$$

$$= \frac{\hbar}{\sqrt{2}} (c_3 + c_2) \underbrace{|\tfrac{1}{2}\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle_2}_{|11\rangle}$$

$$\left.\begin{array}{l} c_2 = -c_3 \\ |c_2|^2 + |c_3|^2 = 1 \end{array}\right\} \quad c_2 = -c_3 = \frac{1}{\sqrt{2}}$$

$$\boxed{|00\rangle = \frac{1}{\sqrt{2}} \left(|\tfrac{1}{2}-\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle_2 - |\tfrac{1}{2}\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2}-\tfrac{1}{2}\rangle_2 \right)}$$

8. Zadata su dva spina odgovarajućim kvantnim brojevima $s_1 = 1$ i $s_2 = \frac{1}{2}$. Naći odgovarajuće koeficijente.

$$s_1 = 1 \qquad\qquad s_2 = \frac{1}{2}$$

$$m_1 = -1, 0, 1 \qquad\qquad m_2 = -\frac{1}{2}, \frac{1}{2}$$

$$\left.\begin{aligned} \hat{\vec{S}}^2 |sm\rangle &= \hbar^2 s(s+1) |sm\rangle \\ \hat{S}_z |sm\rangle &= m\hbar |sm\rangle \end{aligned}\right\} \quad \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} : |sm\rangle$$

$$|sm\rangle = \sum_{s_1, s_2, m_1} C(s_1 s_2\, m_1, m_2 = m - m_1; sm) |s_1 m_1\rangle_1 \otimes |s_2 m_2\rangle_2$$

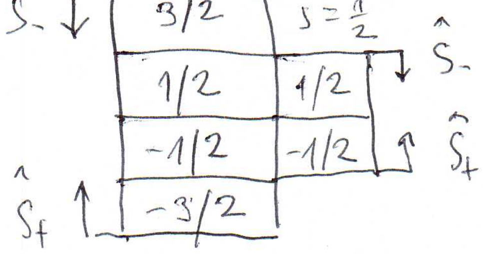
$$s \in \{|s_1 - s_2|, |s_1 - s_2| + 1, \ldots, s_1 + s_2 - 1, s_1 + s_2\} = \left\{\frac{1}{2}, \frac{3}{2}\right\}$$

$$s = \frac{1}{2}, \quad m \doteq -\frac{1}{2}, \frac{1}{2}$$

$$s = \frac{3}{2}, \quad m \doteq -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$$

Standardni bazis za opservablu $\hat{\vec{S}}$ je

$$\{|sm\rangle\} = \left\{ \left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle_u, \left|\tfrac{1}{2}\, -\tfrac{1}{2}\right\rangle_u, \left|\tfrac{3}{2}\tfrac{3}{2}\right\rangle_v, \left|\tfrac{3}{2}\tfrac{1}{2}\right\rangle_v, \left|\tfrac{3}{2}\, -\tfrac{1}{2}\right\rangle_v, \left|\tfrac{3}{2}\, -\tfrac{3}{2}\right\rangle_v \right\}$$



Tenzorski proizvod bazisa

$\mathcal{H}^{(1)}$ $\qquad\qquad$ $\mathcal{H}^{(2)}$

$|11\rangle_1 \, , |10\rangle_1 \, , |1\,-1\rangle_1 \qquad |\tfrac{1}{2}\,\tfrac{1}{2}\rangle_2 \, , |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_2$

pa bazis glasi

$|11\rangle_1 \otimes |\tfrac{1}{2}\,\tfrac{1}{2}\rangle_2$

$|11\rangle_1 \otimes |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_2$

$|10\rangle_1 \otimes |\tfrac{1}{2}\,\tfrac{1}{2}\rangle_2$

$|10\rangle_1 \otimes |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_2$

$|1\,-1\rangle_1 \otimes |\tfrac{1}{2}\,\tfrac{1}{2}\rangle_2$

$|1\,-1\rangle_1 \otimes |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_2$

$$\boxed{|\tfrac{3}{2}\,\tfrac{3}{2}\rangle_u = |11\rangle_1 \otimes |\tfrac{1}{2}\,\tfrac{1}{2}\rangle_2}$$

$$\hat{S}_- |\tfrac{3}{2}\,\tfrac{3}{2}\rangle_u = \left(\hat{S}_{1-} \otimes \hat{I}_2 + \hat{I}_1 \otimes \hat{S}_{2-}\right) |11\rangle_1 \otimes |\tfrac{1}{2}\,\tfrac{1}{2}\rangle_2$$

$$\hbar\sqrt{\tfrac{3}{2}(\tfrac{3}{2}+1) - \tfrac{3}{2}(\tfrac{3}{2}-1)}\; |\tfrac{3}{2}\,\tfrac{1}{2}\rangle = \hat{S}_{1-} |11\rangle_1 \otimes |\tfrac{1}{2}\,\tfrac{1}{2}\rangle_2 + |11\rangle_1 \otimes \hat{S}_{2-} |\tfrac{1}{2}\,\tfrac{1}{2}\rangle \Longrightarrow$$

$$\sqrt{3}\hbar\, |\tfrac{3}{2}\,\tfrac{1}{2}\rangle = \hbar\sqrt{1(1+1) - 1(1-1)}\; |10\rangle_1 \otimes |\tfrac{1}{2}\,\tfrac{1}{2}\rangle_2 + |11\rangle_1 \otimes \hbar\sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1) - \tfrac{1}{2}(\tfrac{1}{2}-1)}\; |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_2$$

$$\sqrt{3}\hbar\, |\tfrac{3}{2}\,\tfrac{1}{2}\rangle = \sqrt{2}\hbar\, |10\rangle_1 \otimes |\tfrac{1}{2}\,\tfrac{1}{2}\rangle_2 + \hbar\, |11\rangle_1 \otimes |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_2$$

$$\boxed{|\tfrac{3}{2}\,\tfrac{1}{2}\rangle_u = \sqrt{\tfrac{2}{3}}\; |10\rangle_1 \otimes |\tfrac{1}{2}\,\tfrac{1}{2}\rangle_2 + \tfrac{1}{\sqrt{3}}\, |11\rangle_1 \otimes |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_2}$$

$$\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle_u = c_1 |11\rangle_1 \otimes \left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle_2 + c_2 |11\rangle_1 \otimes \left|\tfrac{1}{2}-\tfrac{1}{2}\right\rangle_2 +$$

$$c_3 |10\rangle_1 \otimes \left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle_2 + c_4 |10\rangle_1 \otimes \left|\tfrac{1}{2}-\tfrac{1}{2}\right\rangle_2 + c_5 |1-1\rangle_1 \otimes \left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle_2 +$$

$$c_6 |1-1\rangle_1 \otimes \left|\tfrac{1}{2}-\tfrac{1}{2}\right\rangle_2 \quad , \quad c_1 = c_4 = c_5 = c_6 = 0$$

$$\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle_u = c_2 |11\rangle_1 \otimes \left|\tfrac{1}{2}-\tfrac{1}{2}\right\rangle_2 + c_3 |10\rangle_1 \otimes \left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle_2$$

$${}_u\left\langle\tfrac{3}{2}\tfrac{1}{2}\right|\left.\tfrac{1}{2}\tfrac{1}{2}\right\rangle_u = 0$$

$$\left(\sqrt{\tfrac{2}{3}}\ {}_1\langle 10| \otimes {}_2\left\langle\tfrac{1}{2}\tfrac{1}{2}\right| + \frac{1}{\sqrt{3}}\ {}_1\langle 11| \otimes {}_2\left\langle\tfrac{1}{2}-\tfrac{1}{2}\right|\right)\left(c_2 |11\rangle_1 \otimes \left|\tfrac{1}{2}-\tfrac{1}{2}\right\rangle_2 +\right.$$

$$\left. c_3 |10\rangle_1 \otimes \left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle_2 \right) = 0$$

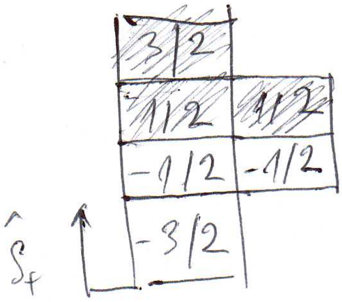
$$\frac{c_2}{\sqrt{3}} + c_3\sqrt{\frac{2}{3}} = 0 \quad \Rightarrow \quad c_2 = -\sqrt{2}\, c_3$$

$$|c_2|^2 + |c_3|^2 = 1$$

$$2|c_3|^2 + |c_3|^2 = 1$$

$$3|c_3|^2 = 1 \quad \Rightarrow \quad c_3 = \frac{1}{\sqrt{3}} \quad \Rightarrow \quad c_2 = -\sqrt{\frac{2}{3}}$$

$$\boxed{\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle_u = -\sqrt{\frac{2}{3}}\ |11\rangle_1 \otimes \left|\tfrac{1}{2}-\tfrac{1}{2}\right\rangle_2 + \frac{1}{\sqrt{3}}\ |10\rangle_1 \otimes \left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle_2}$$



Ostatak za domaći!

$$\underbrace{\left|\tfrac{3}{2}\ -\tfrac{3}{2}\right\rangle_u} = |1\ -1\rangle_1 \otimes \left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle_2$$

$$\hat{S}_+ \left|\tfrac{3}{2}\ -\tfrac{3}{2}\right\rangle_u = \dots \qquad \underbrace{\left|\tfrac{3}{2}\ -\tfrac{1}{2}\right\rangle_u}$$

$$_u\left\langle \tfrac{3}{2}\ -\tfrac{1}{2} \middle| \tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle_u = 0 \Rightarrow \underbrace{\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle_u} = \dots$$

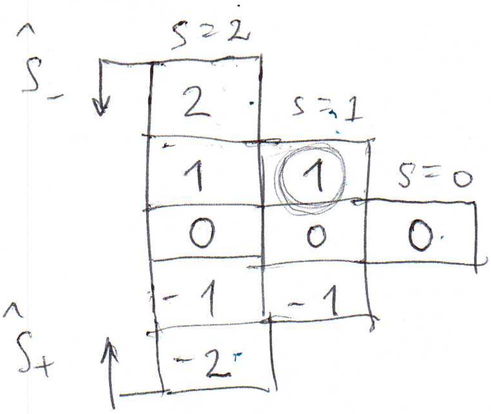
9. Zadata su dva spina $s_1 = s_2 = 1$. Nađi CG koeficijente za stanje $|11\rangle_u$.

$$\hat{\vec{S}}_1 \qquad s_1 = 1 \Rightarrow m_1 = 1, 0, -1 \; : \; |11\rangle_1, |10\rangle_1, |1-1\rangle_1$$

$$\hat{\vec{S}}_2 \qquad s_2 = 1 \Rightarrow m_2 = 1, 0, -1 \; : \; |11\rangle_2, |10\rangle_2, |1-1\rangle_2$$

$$\hat{\vec{S}} \qquad s \in \{ |s_1 - s_2|, \dots s_1 + s_2 \} = \{0, 1, 2\}$$

$$\{|sm\rangle\} = \{ |00\rangle_u, |11\rangle_u, |10\rangle_u, |1-1\rangle_u, |22\rangle_u, |21\rangle_u, |20\rangle_u, |2-1\rangle_u, |2-2\rangle_u \}$$



$$\hat{S}_- = \hat{S}_{1-} \otimes \hat{I}_2 + \hat{I}_1 \otimes \hat{S}_{2-}$$

$$\hat{S}_+ = \hat{S}_{1+} \otimes \hat{I}_2 + \hat{I}_1 \otimes \hat{S}_{2+}$$

$$|22\rangle_u = |11\rangle_1 \otimes |11\rangle_2$$

$$\hat{S}_- |22\rangle_u = \dots \Rightarrow |21\rangle_u = \dots$$

$$_u\langle 11 | 21 \rangle_u = 0 \Rightarrow |11\rangle_u = \dots$$

Domaći!

10. Stanje elektrona u vodonikovom atomu zadato je kvantnim brojevima $|j\,m\rangle = |3/2\ 1/2\rangle$, gde j odgovara opservabli $\hat{\vec{J}} = \hat{\vec{L}} + \hat{\vec{S}}$. Naći verovatnoću da se simultanim merenjem opservabli $\hat{\vec{L}}^2$ i $\hat{L}_z$ u tom stanju, dobiju vrednosti $2\hbar^2$ i $\hbar$, redom.

$$\hat{\vec{L}}^2 |l\,m\rangle = \hbar^2 l(l+1) |l\,m\rangle \Longrightarrow l = 1$$

$$\hat{L}_z |l\,m\rangle = \hbar m |l\,m\rangle \Longrightarrow m = 1$$

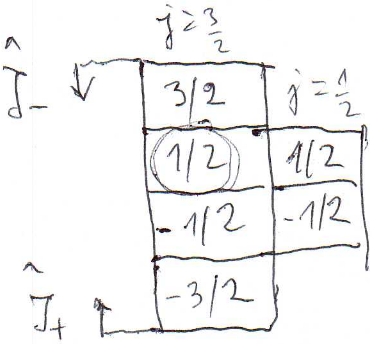
$$|l\,m\rangle_0 = |1\,1\rangle_0$$

$$W(\hat{\vec{L}}^2, \hat{L}_z, |3/2\ 1/2\rangle, 2\hbar^2, \hbar) = {}_U\langle 3/2\ 1/2| \hat{P}_{11} |3/2\ 1/2\rangle_U$$

$$\hat{P}_{11} = |1\,1\rangle_0\langle 1\,1| = \hat{P}_{11} \otimes \hat{I}_S$$

$$\left|\tfrac{3}{2}, \tfrac{1}{2}\right\rangle = ?$$

$j = \frac{3}{2}$ fiksirano, pa su odgovarajući "ormari"



$$\hat{J}_- = \hat{L}_- \otimes \hat{I}_S + \hat{I}_L \otimes \hat{S}_-$$

$$\left|\tfrac{3}{2}\,\tfrac{3}{2}\right\rangle_U = |1\,1\rangle_0 \otimes \left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle_S$$

$$\hat{J}_- \left|\tfrac{3}{2}\,\tfrac{3}{2}\right\rangle_U = \hbar\sqrt{\tfrac{3}{2}\left(\tfrac{3}{2}+1\right) - \tfrac{3}{2}\left(\tfrac{3}{2}-1\right)}\left|\tfrac{3}{2}\ \tfrac{3}{2}-\tfrac{1}{2}\right\rangle_U$$

$$= \sqrt{3}\hbar \left|\tfrac{3}{2}\,\tfrac{1}{2}\right\rangle_U$$

$$\left(\hat{L}_- \otimes \hat{I}_s + \hat{I}_L \otimes \hat{S}_-\right)|11\rangle_0 \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle_s =$$

$$\hat{L}_- \otimes \hat{I}_s\, |11\rangle_0 \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle_s + \hat{I}_L \otimes \hat{S}_-\, |11\rangle_0 \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle_s =$$

$$= \hbar\sqrt{1(1+1)-1(1-1)}\,|10\rangle_0 \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle_s + |11\rangle_0 \otimes \hbar\sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1)-\tfrac{1}{2}(\tfrac{1}{2}-1)}\cdot |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_s$$

$$= \sqrt{2}\hbar\,|10\rangle_0 \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle_s + \hbar\,|11\rangle_0 \otimes |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_s$$

$$\sqrt{3}\hbar\,|\tfrac{3}{2}\tfrac{1}{2}\rangle_0 = \sqrt{2}\hbar\,|10\rangle_0 \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle_s + \hbar\,|11\rangle_0 \otimes |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_s$$

$$|\tfrac{3}{2}\tfrac{1}{2}\rangle_0 = \sqrt{\tfrac{2}{3}}\,|10\rangle_0 \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle_s + \frac{1}{\sqrt{3}}\,|11\rangle_0 \otimes |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_s$$

$$W(\ldots) = \left(\sqrt{\tfrac{2}{3}}\,{}_0\langle 10| \otimes {}_s\langle\tfrac{1}{2}\tfrac{1}{2}| + \frac{1}{\sqrt{3}}\,{}_0\langle 11| \otimes {}_s\langle\tfrac{1}{2}\,-\tfrac{1}{2}|\right)$$

$$|11\rangle_0\langle 11| \otimes \hat{I}_s \left(\sqrt{\tfrac{2}{3}}\,|10\rangle_0 \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle_s + \frac{1}{\sqrt{3}}\,|11\rangle_0 \otimes |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_s\right)$$

$$= \frac{1}{3}$$

Za pismeni varijanta sa a) $|jm\rangle \equiv |\tfrac{3}{2}\,-\tfrac{1}{2}\rangle$

) $|jm\rangle = |\tfrac{5}{2}\,\tfrac{3}{2}\rangle$

$j=5/2$	$j=3/2$	$j=1/2$
5/2		
3/2	3/2	
1/2	1/2	1/2
-1/2	-1/2	-1/2
-3/2	-3/2	
-5/2		

11. Sistem od dva spina zadat kv. brojevima $s_1 = 1,\ s_2 = 2$ nalazi se u stanju $|sm\rangle_u$

a) $|32\rangle_u$, $\boxed{b)\ |3\,-2\rangle_u}$ Za ispit!

Naći verovatnoću da se simultanim merenjem opservabli $\hat{S}_{1z}, \hat{S}_{2z}$ dobiju vrednosti $\hbar, \hbar$.

---

a) $\hat{\vec{S}}_1$, $s_1 = 1$, $m_1 = 1, 0, -1$

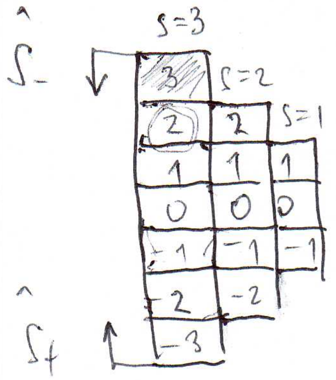
$\hat{\vec{S}}_2$, $s_2 = 2$, $m_2 = -2, -1, 0, 1, 2$

Bazisi

$\mathcal{H}^{(1)}$: $|11\rangle_1, |10\rangle_1, |1\,-1\rangle_1$

$\mathcal{H}^{(2)}$: $|22\rangle_2, |21\rangle_2, |20\rangle_2, |2\,-1\rangle_2, |2\,-2\rangle_2$

$\hat{\vec{S}} = \hat{\vec{S}}_1 + \hat{\vec{S}}_2$ $\quad s \in \{|s_1 - s_2|, \dots s_1 + s_2\} = \{1, 2, 3\}$



$\hat{S}_{1z} |sm\rangle_1 = m\hbar |sm\rangle_1 \quad \Rightarrow m_1 = 1 \quad , s_1 = 1$

$\hat{S}_{2z} |sm\rangle_2 = m\hbar |sm\rangle_2 \quad \Rightarrow m_2 = 1 \quad , s_2 = 2$

$\hat{P}_1 = |11\rangle_1 \langle 11| \qquad \hat{P}_2 = |21\rangle_2 \langle 21|$

$\hat{P} = \hat{P}_1 \otimes \hat{P}_2$

$|33\rangle_U = |11\rangle_1 \otimes |22\rangle_2$

$\hat{S}_- |33\rangle_U = \hbar\sqrt{6}\, |32\rangle_u$; $\hat{S}_- = \hat{S}_{1-} \otimes \hat{I}_2 + \hat{I}_1 \otimes \hat{S}_{2-}$

$(\hat{S}_{1-} \otimes \hat{I}_2 + \hat{I}_1 \otimes \hat{S}_{2-})\, |11\rangle_1 \otimes |22\rangle_2 =$

$\hat{S}_{1-} |11\rangle_1 \otimes |22\rangle_2 + |11\rangle_1 \otimes \hat{S}_{2-} |22\rangle_2 =$

$\hbar\sqrt{2}\, |10\rangle_1 \otimes |22\rangle_2 + \hbar\sqrt{4}\, |11\rangle_1 \otimes |21\rangle_2$

Jednačedi talasasto podvučeno

$\hbar\sqrt{6}\, |32\rangle_u = \hbar\sqrt{2}\, |10\rangle_1 \otimes |22\rangle_2 + \hbar\sqrt{4}\, |11\rangle_1 \otimes |21\rangle_2$

$$\boxed{|32\rangle_u = \frac{1}{\sqrt{3}} |10\rangle_1 \otimes |22\rangle_2 + \sqrt{\frac{2}{3}}\, |11\rangle_1 \otimes |21\rangle_2}$$

$W(\hat{S}_{1z}, \hat{S}_{2z}, |32\rangle_U, \hbar, \hbar) = {}_U\langle 32| \hat{P} |32\rangle_U$

$= {}_U\langle 32| \hat{P}_1 \otimes \hat{P}_2 |32\rangle_U = {}_U\langle 32|11\rangle_1 \langle 11| \otimes |21\rangle_2 \langle 21|32\rangle_U$

$= |{}_1\langle 11|{}_2\langle 21|32\rangle_U|^2 = \frac{2}{3}$

b) Krenuti od

$|3\,{-3}\rangle_U = |1\,{-1}\rangle_1 \otimes |2\,{-2}\rangle_2$

i koristiti operator $\hat{S}_+$

Varijanta!

Umesto $|3\,{-2}\rangle_U$ da bude $|3\,{-1}\rangle_U$ ili $|30\rangle_U$

12. Stanje elektrona u vodonikovom atomu je $|l=2, m_l=1\rangle \otimes |\frac{1}{2}\, -\frac{1}{2}\rangle$. Naći verovatnoću da se merenjem kvadrata opservable $\hat{\vec{J}} = \hat{\vec{L}} + \hat{\vec{S}}$ dobije vrednost a) $j=5/2$ | b) $j=3/2$ Za ispit.

Stanje e^- $\quad |21\rangle_o \otimes |\frac{1}{2}\, -\frac{1}{2}\rangle_s$

$$W\left(\hat{\vec{J}}^2, j=\tfrac{5}{2}, |21\rangle_o \otimes |\tfrac{1}{2}\, -\tfrac{1}{2}\rangle_s\right) =$$

$$\left| {}_u\langle \tfrac{5}{2}, m_j | 21\rangle_o \otimes |\tfrac{1}{2}\, -\tfrac{1}{2}\rangle_s \right|^2 = \left| {}_u\langle \tfrac{5}{2}, \tfrac{1}{2} | 21\rangle_o \otimes |\tfrac{1}{2}\, -\tfrac{1}{2}\rangle \right|^2$$

$$|\tfrac{5}{2}, \tfrac{1}{2}\rangle_u = ?$$

$j=5/2$	$j=3/2$	$j=1/2$
5/2		
3/2	3/2	
1/2	1/2	1/2
−1/2	−1/2	−1/2
−3/2	−3/2	
−5/2		

Kako je u pitanju primer sa orbitalnim i spinskim stepenima slobode zar nije bolje
$\hat{J}_- = \hat{L}_- \otimes \hat{I}_s + \hat{I}_L \otimes \hat{S}_-$
?

$$|\tfrac{5}{2}\, \tfrac{5}{2}\rangle_u = |22\rangle_o \otimes |\tfrac{1}{2}\, \tfrac{1}{2}\rangle_s$$

Dva puta se deluje operatorom $\hat{S}_- = \hat{S}_{-o} \otimes \hat{I}_s + \hat{I}_o \otimes \hat{S}_{-s}$

$$\hat{S}_- |\tfrac{5}{2}\, \tfrac{5}{2}\rangle_u = \hbar\sqrt{5}\, |\tfrac{5}{2}\, \tfrac{3}{2}\rangle_u$$

$$\hat{S}_-^2 |\tfrac{5}{2}\, \tfrac{5}{2}\rangle_u = \hbar\sqrt{5}\, \hat{S}_- |\tfrac{5}{2}\, \tfrac{3}{2}\rangle_u = \hbar^2 \sqrt{5}\sqrt{8}\, |\tfrac{5}{2}\, \tfrac{1}{2}\rangle$$

$$= \hbar^2 \sqrt{40}\, |\tfrac{5}{2}\, \tfrac{1}{2}\rangle$$

$$\left(\hat{S}_{-0} \otimes \hat{I}_s + \hat{I}_0 \otimes \hat{S}_{-s}\right) |22\rangle_0 \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle_s$$

$$\hat{S}_{-0} \otimes I_s |22\rangle_0 \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle_s + \hat{I}_0 \otimes \hat{S}_{-s} |22\rangle_0 \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle_s$$

$$\hbar\sqrt{4}\, |21\rangle_0 \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle_s + |22\rangle_0 \otimes \hbar |\tfrac{1}{2}\, -\tfrac{1}{2}\rangle_s$$

Opet delovanje

$$\left(\hat{S}_{-0} \otimes \hat{I}_s + \hat{I}_0 \otimes \hat{S}_{-s}\right)\left(\hbar\sqrt{4}\, |21\rangle_0 \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle_s + \right.$$

$$\left.\hbar |22\rangle_0 \otimes |\tfrac{1}{2}\, -\tfrac{1}{2}\rangle_s\right) = \hbar\sqrt{4}\, \hat{S}_{-0} |21\rangle_0 \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle_s$$

$$+ \hbar\, \hat{S}_{-0} |22\rangle_0 \otimes |\tfrac{1}{2}\, -\tfrac{1}{2}\rangle_s + \hbar\sqrt{4}\, |21\rangle_0 \otimes \hat{S}_{-s} |\tfrac{1}{2}\tfrac{1}{2}\rangle_s +$$

$$\hbar |22\rangle_0 \otimes \cancel{\hat{S}_{-s}} |\tfrac{1}{2}\, -\tfrac{1}{2}\rangle_s = \hbar^2\sqrt{4}\sqrt{6}\, |20\rangle_0 \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle_s +$$

$$\hbar^2\sqrt{4}\, |21\rangle_0 \otimes |\tfrac{1}{2}\, -\tfrac{1}{2}\rangle_s + \hbar^2\sqrt{4}\, |21\rangle_0 \otimes |\tfrac{1}{2}\, -\tfrac{1}{2}\rangle_s$$

Jednačeći desne strane

$$\hbar^2\sqrt{40}\, |\tfrac{5}{2}\tfrac{1}{2}\rangle_u = \hbar^2\sqrt{4}\sqrt{6}\, |20\rangle_0 \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle_s +$$

$$2\hbar^2\sqrt{4}\, |21\rangle_0 \otimes |\tfrac{1}{2}\, -\tfrac{1}{2}\rangle_s \implies$$

$$|\tfrac{5}{2}\tfrac{1}{2}\rangle_u = \frac{\sqrt{4}\sqrt{6}}{\sqrt{40}} |20\rangle_0 \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle +$$

$$\frac{2\sqrt{4}}{\sqrt{40}} |21\rangle_0 \otimes |\tfrac{1}{2}\, -\tfrac{1}{2}\rangle_s$$, odnosno

$$|\tfrac{5}{2}\tfrac{1}{2}\rangle_u = \sqrt{\tfrac{3}{5}}\, |20\rangle_0 \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle_s + \sqrt{\tfrac{2}{5}}\, |21\rangle_0 \otimes |\tfrac{1}{2}\, -\tfrac{1}{2}\rangle_s$$

$$W(\ldots) = \frac{2}{5}$$

13. U spinskom faktor prostoru helijuma zadato je stanje elektronskog para:

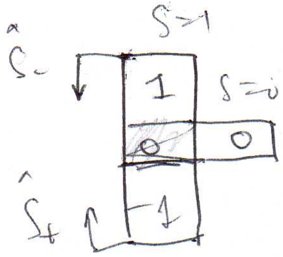
$$|\tfrac{1}{2}\,\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_2 .$$

Naći verovatnoću da se merenjem kvadrata opservable $\vec{S} = \vec{S}_1 + \vec{S}_2$ dobije vrednost $s = 1$.

$$W\left(\vec{S}^2,\ s=1,\ |\tfrac{1}{2}\,\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_2\right) = ?$$

$$\sum_{m_s} \left|{}_u\langle 1\, m_s | \tfrac{1}{2}\,\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_2\right|^2 =$$

$$\left|{}_u\langle 1\,0 | \tfrac{1}{2}\,\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_2\right|^2$$



$$|1\,1\rangle_u = |\tfrac{1}{2}\,\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2}\,\tfrac{1}{2}\rangle_2$$

$$\hat{S}_- |1\,1\rangle_u = \hbar\sqrt{1(1+1)} = \sqrt{2}\,\hbar\,|1\,0\rangle_u$$

$$\hat{S}_- = (\hat{S}_{1-} + \hat{S}_{2-})$$

$$(\hat{S}_{1-} + \hat{S}_{2-})\,|\tfrac{1}{2}\,\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2}\,\tfrac{1}{2}\rangle_2 =$$

$$\hat{S}_{1-}|\tfrac{1}{2}\,\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2}\,\tfrac{1}{2}\rangle_2 + \hat{S}_{2-}|\tfrac{1}{2}\,\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2}\,\tfrac{1}{2}\rangle_2$$

$k\left|\frac{1}{2}\,-\frac{1}{2}\right\rangle_1 \otimes \left|\frac{1}{2}\,\frac{1}{2}\right\rangle_2 + k\left|\frac{1}{2}\,\frac{1}{2}\right\rangle_1 \otimes \left|\frac{1}{2}\,-\frac{1}{2}\right\rangle_2$

$$|10\rangle_U = \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}\,-\frac{1}{2}\right\rangle_1 \otimes \left|\frac{1}{2}\,\frac{1}{2}\right\rangle + \left|\frac{1}{2}\,\frac{1}{2}\right\rangle_1 \otimes \left|\frac{1}{2}\,-\frac{1}{2}\right\rangle_2\right)$$

Jasno je da je tražena verovatnoća

$W(\dots) = \frac{1}{2}$

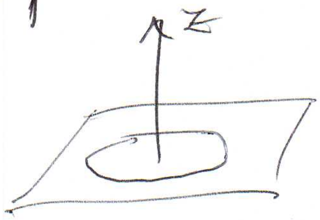
4. Rešiti stacionarnu Šredingerovu j-nu za sledeće fizičke sisteme:

a) Rotator u ravni

b) Rotator u prostoru

---

a)



Sistem koji rotira u ravni oko z ose

$$\hat{H} = \frac{\hat{L}_z^2}{2I}$$

$\hat{L}_z$ - z-komponenta momenta impulsa

I - moment inercije

$$\hat{H}|\psi_m\rangle = E_m|\psi_m\rangle$$

Sa druge strane $\hat{L}_z|\psi_m\rangle = m\hbar|\psi_m\rangle$

$$\hat{L}_z^2|\psi_m\rangle = m^2\hbar^2|\psi_m\rangle$$

$[\hat{H},\hat{L}_z]=0 \Rightarrow |\psi_m\rangle$ zajedničko

$$E_m = \frac{m^2\hbar^2}{2I}$$

Pored $\hat{H}, \hat{L}_z$ tu je i $\hat{\vec{L}}^2$ — čine PSKO

Zajedničko svojstveno stanje $|\psi_{\ell m}\rangle \xrightarrow{\vec{r}\text{-reprezentacija}} Y_\ell(\theta,\varphi)$

Sferni harmon

b) $\hat{H} = \dfrac{\hat{\vec{L}}^2}{2I}$

$\hat{\vec{L}}^2 |\psi_{lm}\rangle = l(l+1)\hbar^2 |\psi_{lm}\rangle$

$[\hat{H}, \hat{\vec{L}}^2] = 0 \Rightarrow |\psi_{lm}\rangle$ zajedničko

$\hat{H} |\psi_{lm}\rangle = E_{lm} |\psi_{lm}\rangle$

$E_{lm} = \dfrac{l(l+1)\hbar^2}{2I}$ – sv. vrednosti

$|\psi_{lm}\rangle \equiv |lm\rangle \xrightarrow{\vec{r}\text{-reprez.}} Y_l^m(\theta, \varphi)$ – sv. f-je.